Internal friction between solitons in near-integrable systems

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We study inelastic interactions (internal friction) between two solitons in a system described by a near-integrable generalized nonlinear Schrödinger equation. An analytic method for calculating the radiation intensity and the rates of amplitude changes of two interacting solitons is proposed. This method shows that, in the limit of zero angle of collision, the amplitude of the smaller soliton decays so that it is inversely proportional to the cube root of the propagation distance. The radiation losses associated with the internal friction of two solitons occur at the expense of the smaller soliton. Half of the energy lost from the smaller soliton is radiated and half goes to the larger soliton. The results obtained by use of our analytic method are in excellent agreement with numerical simulations.

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I. INTRODUCTION

The nonlinear Schrödinger equation (NLSE) plays an important role in many physical problems [1,2]. It arises in various fields (e.g., in optics [3,4], theory of waves on deep water [5], quantum field theory [6], and so on). One of the main features of the NLSE is the existence of a special class of localized solutions, solitons, which are robust against perturbations and demonstrate a particlelike behavior. These unique features of solitons can be used in various physical applications. For example, in optics solitons are expected to be suitable information carriers in optical communication systems [3].

The NLSE is integrable by the inverse scattering technique [7]. The inverse scattering technique shows that the radiative and soliton parts of any solution of the NLSE are noninteracting. They may form a nonlinear superposition, but they do not exchange the energy and do not transform into each other. Solitons themselves can also form a nonlinear superposition, but do not mix their energies. In general, the interactions of solitons in any integrable system (e.g., those described by the Korteveg–de Vries [2] or the NLSE equations) are elastic. This means that the interaction (collision) leaves the shapes of solitons unaltered (it only creates phase shifts). The interaction is elastic due to the fact that the associated equations possess an infinite number of conserved quantities (this is often used as a definition of integrability). In contrast, it has been shown numerically in a number of papers [8–10] that the interaction of solitons in systems described by nonintegrable equations leads to additional radiation being emitted from the impact area of the soliton interaction. An analytical study of the inelastic interaction for the Peregrine-Benjamin equation has been made by Kodama [11]. Reconstruction of two-soliton solutions of the generalized NLSE has been considered in [12]. However, the rate of radiation and the details of the interaction between the solitonlike solutions in nonintegrable systems is still an open question.

In this paper we consider two-soliton interactions in a particular system (a "near-Kerr" system) described by an equation which is close to the NLSE. We will formulate and solve an optical problem, but due to various possible applications of the NLSE, our results can be extended to other fields of physics.

The remainder of the paper is organized as follows. In Sec. II we discuss the subject of our analysis (i.e., give a definition of a *near-Kerr* system) and formulate the problem. In Sec. III we develop a method for calculating the radiation intensity and derive an asymptotic analytic expression for amplitude changes versus propagation distance of two copropagating solitons in a near-Kerr system. In Sec. IV we present numerical results and compare them with our analytic results. We also compare our results qualitatively with the results obtained recently for the breather interactions in nonlinear lattices [13] and show the generality of these new phenomena. Section V contains conclusions and a discussion of possible applications.

II. STATEMENT OF THE PROBLEM

The generalized equation, describing picosecond pulse propagation in nonlinear optical fiber, is [3]

$$i\frac{\partial U}{\partial \xi}+\frac{\partial^2 U}{\partial \tau^2}+f(|U|^2)U=0\ , \tag{1}$$

where $U(\xi,\tau)$ is the slowly varying electric field envelope, ξ is the normalized longitudinal coordinate, τ is the normalized retarded time, and the form of the function $f(|U|^2)$ depends on the nonlinearity of the fiber material refractive index. In this paper we consider the case $f(|U|^2)=|U|^2+\alpha|U|^4$:

$$i\frac{\partial U}{\partial \xi}+\frac{\partial^2 U}{\partial \tau^2}+(|U|^2+\alpha|U|^4)U=0\ , \tag{2}$$

where α is the normalized fifth-order nonlinear susceptibility. The Kerr-law nonlinearity occurs when $\alpha=0$. In this case we have the conventional NLSE, which is integrable by means of an inverse scattering transform. If we

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want to investigate the system which is close to NLSE, then we require the condition $|\alpha||U|^2 \ll 1$ to be satisfied.

The single solitonlike solution of Eq. (2) can be obtained for any value of α:

$$U(\xi,\tau)=\frac{2\eta\exp(i\eta^2\xi)}{[1+\cosh(2\eta\tau)(1+\frac{16}{3}\alpha\eta^2)^{1/2}]^{1/2}}\ , \tag{3}$$

where η is the soliton parameter (we assume that $\eta>0$ without loss of generality). In the limit $\alpha=0$, the solution (3) reduces to the NLSE soliton $U(\xi,\tau)=\sqrt{2}\eta\,\mathrm{sech}(\eta\tau)\exp(i\eta^2\xi)$.

The pulse energy $Q=\int_{-\infty}^{\infty}|U|^2d\tau$ carried by a solitonlike solution (3) is

$$Q=\frac{2\sqrt{3}}{\sqrt{\alpha}}\arctan\left[\frac{\sqrt{16\alpha}\eta}{\sqrt{3+3+16\eta^2\alpha}}\right] \tag{4}$$

in the case of $\alpha>0$ and

$$Q=\frac{\sqrt{3}}{\sqrt{-\alpha}}\ln\left[\frac{\sqrt{3}+\sqrt{3+16\eta^2\alpha}+\sqrt{-16\eta^2\alpha}}{\sqrt{3}+\sqrt{3+16\eta^2\alpha}-\sqrt{-16\eta^2\alpha}}\right] \tag{5}$$

in the case of $-3/16\eta^2<\alpha<0$. In each case the result for $|\alpha|\eta^2 \ll 1$ is

$$Q\approx 4\eta\left[1-\frac{4\alpha\eta^2}{3}+\cdots\right]. \tag{6}$$

This expression reduces to $Q=4\eta$ (the result for NLSE) when $\alpha=0$ and shows explicitly the small parameter of our problem. This parameter

$$\gamma=\frac{4\alpha\eta^2}{3} \tag{7}$$

gives a quantitative definition of a near-Kerr system [we call the system (2) a near-Kerr system if $|\gamma|\ll 1$]. Below we will analyze two-soliton interactions described by the near-Kerr Eq. (2).

The system described by the near-Kerr Eq. (2) belongs to a specific class of systems which are close to integrable. In optics this system is important from a practical point of view, since soliton propagation in real nonlinear optical fibers can be described by equations which are close to, but not exactly, integrable. To understand the processes which are occurring in the impact area of a two-soliton collision in more detail, we start our analysis with the case of the zero angle of collision (i.e., copropagation of two solitons). The main attention will be paid to the radiation which is emitted from the area of interaction.

III. TWO COPROPAGATING SOLITONS IN A NEAR-KERR SYSTEM

For the NLSE [Eq. (2) with $\alpha=0$], two-soliton (TS) solutions are well known [14]:

$$U_{\mathrm{TS}}(\xi,\tau)=\frac{2\sqrt{2}(\eta_1^2-\eta_2^2)\{\eta_2\cosh(\eta_1\tau)+\eta_1\cosh(\eta_2\tau)\exp[i(\eta_1^2-\eta_2^2)\xi]\}\exp(i\eta_2^2\xi)}{(\eta_1-\eta_2)^2\cosh[(\eta_1+\eta_2)\tau]+(\eta_1+\eta_2)^2\cosh[(\eta_1-\eta_2)\tau]+4\eta_1\eta_2\cos[(\eta_1^2-\eta_2^2)\xi]}\ . \tag{8}$$

[A particular case of the solution (8) with $\eta_1=1$ and $\eta_2=3$ has been found by Satsuma and Yajima [15].] The solution (8) is the result of the nonlinear interference of two NLSE solitons

$$U_1=\frac{\sqrt{2}\eta_1\exp(i\eta_1^2\xi)}{\cosh(\eta_1\tau)}\ ,\quad U_2=\frac{\sqrt{2}\eta_2\exp(i\eta_2^2\xi)}{\cosh(\eta_2\tau)}\ , \tag{9}$$

with the amplitudes $A_1=\sqrt{2}\eta_1$ and $A_2=\sqrt{2}\eta_2$. The center of each soliton is located at $\tau=0$. When $\alpha\neq 0$ in Eq. (2), formally the solution (8) no longer holds. However, we can assume that the solution of a near-Kerr problem still can be represented, to good accuracy, as the beats of two solitons of the NLSE [i.e., by the solution (8)]. Due to non-integrability, there is also radiation from the interaction area. Thus the form of the generalized two-soliton solution of Eq. (2) (with $\alpha\neq 0$) is the

$$U(\xi,\tau)=U_{\mathrm{TS}}(\xi,\tau)+f_{\mathrm{rad}}(\xi,\tau)\ , \tag{10}$$

where $U_{\mathrm{TS}}(\xi,\tau)$ is the two-soliton solution (8) of the conventional NLSE and f_{rad} is the radiation part. Two interacting solitons are located in a finite area around the origin of the τ-coordinate axis, so that asymptotically (i.e., in the limits $\tau\to\pm\infty$) the radiation can be represented as a superposition of linear plane waves:

$$f_{\mathrm{rad}}(\xi,\tau\to\pm\infty)=\sum_n a_n e^{i(\omega_n\xi\pm\sqrt{-\omega_n}\tau)}\ , \tag{11}$$

where ω_n ($\omega_n<0$) and a_n are, respectively, the frequencies in ξ and amplitudes of the radiation waves.

To determine the energy flow from the two-soliton solution, we consider the invariants of Eq. (2). For our case, the two most important of them are

$$Q=\int_{-\infty}^{\infty}|U|^2dt \tag{12}$$

(the energy) and

$$H=\int_{-\infty}^{\infty}\left[|U_\tau|^2-\frac{|U|^4}{2}-\alpha\frac{|U|^6}{3}\right]dt \tag{13}$$

(the Hamiltonian). These invariants follow from the more general continuity equations

$$\frac{\partial}{\partial\xi}\rho_Q=-\frac{\partial}{\partial\tau}j_Q\ ,\quad \frac{\partial}{\partial\xi}\rho_H=-\frac{\partial}{\partial\tau}j_H\ , \tag{14}$$

where $\rho_Q=|U|^2$ and $\rho_H=|U_\tau|^2-|U|^4/2-\alpha(|U|^6/3)$ are the energy and the Hamiltonian densities and $j_Q=i[(\partial U^*/\partial\tau)U-(\partial U/\partial\tau)U^*]$ and $j_H=-[(\partial U^*/\partial\tau)(\partial U/\partial\xi)+(\partial U/\partial\tau)(\partial U^*/\partial\xi)]$ are the energy and the Hamiltonian flows, respectively. Integration of Eqs. (14) over the interval $(-\tau_0,\tau_0)$, which includes the two-soliton part of the general solution (10) (see Fig. 1), gives the equations for the conservation of energy and the Hamiltonian in the integral form

$$\frac{\partial}{\partial x}Q_{\rm TS} = -[j_Q(\tau_0) - j_Q(-\tau_0)] ,$$
$$\frac{\partial}{\partial x}H_{\rm TS} = -[j_H(\tau_0) - j_H(-\tau_0)] , \tag{15}$$

where $H_{\rm TS}$ and $Q_{\rm TS}$ are defined by the integrals (13) and (12), but with the limits replaced by $-\tau_0$ and τ_0 (i.e., $H_{\rm TS}$ and $Q_{\rm TS}$ are the Hamiltonian and the energy of the two-soliton-like part of the solution). Equations (15) determine the rate of change of $H_{\rm TS}$ and $Q_{\rm TS}$ in terms of the energy and Hamiltonian flows in the points outside of the two-soliton solution where only the radiation part of the solution (10) exists. This radiation part can be presented as a Fourier series of linear waves (11). Substituting expression (11) into (15) and taking into account only waves which are moving away from the two-soliton part of the solution finally gives

$$\frac{\partial Q_{\rm TS}}{\partial \xi} = -4\sum_n \sqrt{-\omega_n}\,|a_n|^2 ,$$
$$\frac{\partial H_{\rm TS}}{\partial \xi} = 4\sum_n \omega_n\sqrt{-\omega_n}\,|a_n|^2 . \tag{16}$$

To solve the system (16), one has to express all unknown quantities in terms of two basic variables. The most convenient variables are the parameters of the two partial solitons η_1 and η_2. We assume that they are changing adiabatically in ξ. Hence, for a near-Kerr system, the energy $Q_{\rm TS}$ and the Hamiltonian $H_{\rm TS}$ can be expressed approximately in terms of η_1 and η_2 in the same form as the two-soliton solution (8) of NLSE:

$$Q_{\rm TS} = 4(\eta_1+\eta_2) , \quad H_{\rm TS} = -\tfrac{4}{3}(\eta_1^3+\eta_2^3) . \tag{17}$$

Now we need to find $\omega_n(\eta_1,\eta_2)$ and $a_n(\eta_1,\eta_2)$. To do this, we use some simplifying assumptions. We assume the existence of two small parameters $|\gamma| \ll 1$ and $\varepsilon = \eta_2/\eta_1 \ll 1$. Thus, in addition to the near-Kerr condition, we assume that the amplitude A_1 of one (the first) partial soliton is much bigger than the amplitude A_2 of the other (the second) partial soliton.

According to expression (11), the frequencies ω_n of radiation are determined by the *negative* frequencies of a source of radiation, i.e., by the negative frequencies in ξ of the two-soliton solution (8). These frequencies are given by

$$\omega_n = \eta_2^2 - n(\eta_1^2-\eta_2^2) , \tag{18}$$

where $n = 1,2,3,4,\ldots$. Expanding the solution (8) in a Fourier series in ξ we find that the component at the lowest frequency $\omega_1 = -(\eta_1^2 - 2\eta_2^2)$ has the largest amplitude. It is possible to conclude that most of the radiation is emitted at this lowest frequency. Indeed, we have found numerically that if $\varepsilon = \eta_2/\eta_1 < 1/4$, then the last statement is correct and the radiation at all higher frequencies (ω_n) can be ignored. Therefore, henceforth we take into account only radiative waves at the ω_1 frequency.

The amplitudes a_n of the plane radiation waves can be obtained approximately using the existence of the small parameters γ and ε (perturbation approach). The radiation appears because of the last term in Eq. (2), which is proportional to γ. Therefore, in the linear approximation, the amplitudes of the radiation waves must also be proportional to γ. Substituting the general solution (10) into Eq. (2), and keeping only terms which are linear in γ, we obtain the equation

$$i\frac{\partial f_{\rm rad}}{\partial \xi} + \frac{\partial^2 f_{\rm rad}}{\partial \tau^2} + F(\xi,\tau)f_{\rm rad} + G(\xi,\tau)f^*_{\rm rad} = -\alpha P(\xi,\tau) , \tag{19}$$

where $F(\xi,\tau) = 2|U_{\rm TS}|^2$, $G(\xi,\tau) = U_{\rm TS}^2$, $P(\xi,\tau) = |U_{\rm TS}|^4 U_{\rm TS}$, and $f^*_{\rm rad}$ is the complex conjugate of $f_{\rm rad}$. Due to adiabaticity of η_1 and η_2 changes, each of the functions $F(\xi,\tau)$, $G(\xi,\tau)$, and $P(\xi,\tau)$ can be considered to be periodic in ξ and can be presented in the form of an infinite Fourier series

$$F(\xi,\tau) = \sum_{n=-\infty}^{\infty} F_n(\tau)e^{i(\eta_2^2-\eta_1^2)n\xi} ,$$
$$G(\xi,\tau) = e^{i2\eta_2^2\xi}\sum_{n=-\infty}^{\infty} G_n(\tau)e^{i(\eta_2^2-\eta_1^2)n\xi} , \tag{20a}$$
$$P(\xi,\tau) = e^{i\eta_2^2\xi}\sum_{n=-\infty}^{\infty} P_n(\tau)e^{i(\eta_2^2-\eta_1^2)n\xi} .$$

We also expand $f_{\rm rad}(\xi,\tau)$ in a Fourier series in ξ, where

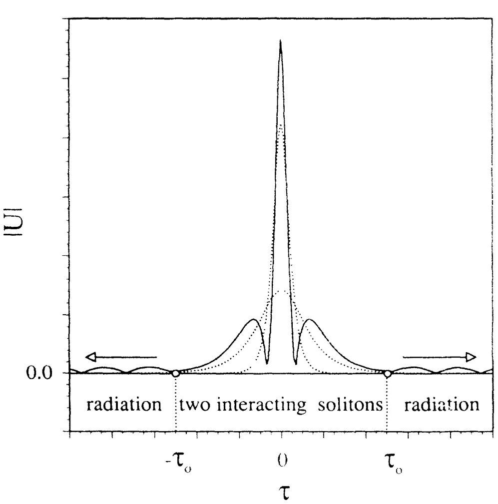

FIG. 1. Schematic plot of the solution (10), which shows the two-soliton part and radiation emitted from it. The points $\tau = -\tau_0$ and τ_0 are the boundaries separating the interacting solitons from the radiation. The soliton part of the solution (10) is formed by a dynamic superposition of two partial solitons, which are shown by dashed curves.

we only retain negative frequencies ω_n, which correspond to radiation:

$$f_{\rm rad}(\xi,\tau)=e^{i\eta_2^2\xi}\sum_{n=1}^{\infty} g_n(\tau)e^{i(\eta_2^2-\eta_1^2)n\xi} . \qquad (20b)$$

After substituting Eqs. (20) into Eq. (19) and collecting the terms in front of the factors $e^{i\eta_2^2\xi}e^{i(\eta_2^2-\eta_1^2)n\xi}$, one can get an infinite set of equations

$$\frac{\partial^2 g_n}{\partial\tau^2}-\omega_n g_n+\sum_{j=1}^{\infty}F_{n-j}(\tau)g_j(\tau)+\sum_{j=1}^{\infty}G_{n+j}(\tau)g_j^*(\tau) = -\alpha P_n(\tau) , \qquad (21)$$

where $1\leq n<\infty$. This infinite set of ordinary differential equations (21) defines the radiation at various frequencies ω_n. The terms on the right-hand side of Eq. (21) can be considered as source terms. As we explained above, we are interested in radiation at the frequency ω_1, and thus need to find only $g_1(\tau)$. It is possible to separate the equation for $g_1(\tau)$ from all the others using our perturbation approach. Making the substitution $g_1(\tau)=\alpha\varepsilon^2\eta_1^3 b(t)$ (where $t=\tau\eta_1$) in Eq. (21) (with $n=1$) and keeping terms up to order $\gamma\varepsilon^2$, one can obtain the following equation for $b(t)$:

$$\frac{\partial^2 b}{\partial t^2}+b+\frac{4b}{\cosh^2 t}=-\frac{4\sqrt{2}}{\cosh^3 t}+\frac{24\sqrt{2}}{\cosh^5\tau}-\frac{24\sqrt{2}}{\cosh^7\tau} . \qquad (22)$$

Equation (22) can be solved either analytically (see Appendix A) or numerically. One of the the solutions of Eq. (22) (corresponding to radiation) is shown in Fig. 2. (The other one is the mirror image of the solution in Fig. 2 with respect to $t=0$ axis). The amplitude of the oscillations in these solutions, in the limit $t=\infty$ or $-\infty$ is $b_0\approx 1.987$ (the exact value of b_0 is given in Appendix A). Thus the amplitude of radiation at the main radiation frequency ω_1 is given by

$$a_1=\alpha\eta_2^2\eta_1 b_0 \qquad (23)$$

up to the first order in γ and up to the second order in ε

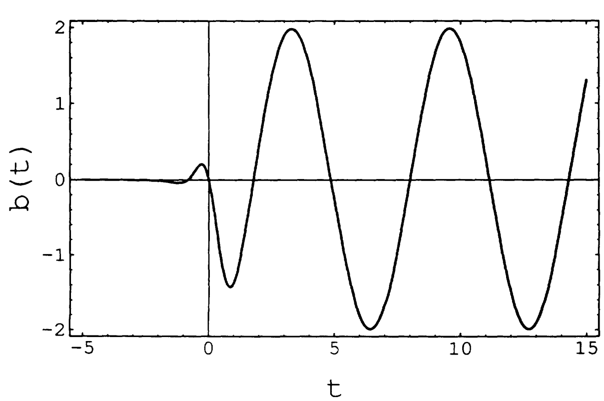

FIG. 2. Solution of Eq. (22) with the asymptotic boundary conditions $b(t=-\infty)=0$ and $(\partial b/\partial t)(t=-\infty)=0$. [$b(t)$ and t are dimensionless variables.]

of our perturbation approach.

Equations (16), after neglecting radiation at frequencies different from ω_1 and substituting expressions (17), (18), and (23), have the form

$$\begin{aligned}\frac{\partial(\eta_1+\eta_2)}{\partial\xi}&=-\alpha^2 b_0^2\sqrt{n_1^2-2\eta_2^2}\eta_1^2\eta_2^4 ,\\ \frac{\partial(\eta_1^3+\eta_2^3)}{\partial\xi}&=3\alpha^2 b_0^2(\sqrt{\eta_1^2-2\eta_2^2})^3\eta_1^2\eta_2^4 .\end{aligned} \qquad (24)$$

Equations (24) have the first integral

$$2\eta_1(\xi)+\eta_2(\xi)=K , \qquad (25)$$

where $K=2\eta_1(0)+\eta_2(0)$. The expression (25) gives a relative dynamics of the soliton parameters of two interacting solitons. Substituting (25) into any equation of the system (24) one can get a solution in quadratures:

$$\xi=\frac{4}{\alpha^2 b_0^2 K^6}\int_{\eta_2(\xi)/K}^{\eta_2(0)/K}\frac{dy}{y^4(1-y)^2\sqrt{1-2y-7y^2}} . \qquad (26)$$

The integral in the expression (26) can be exactly evaluated in terms of elementary functions to give an implicit relationship between ξ and η_2. However, since we have obtained Eqs. (24) in the approximation $\varepsilon=\eta_2/\eta_1\ll 1$ we can use this small parameter again in the approximate calculation of the integral in Eq. (26) to get a simple asymptotic result

$$\eta_2(\xi)=\frac{1}{[\beta\xi+\eta_2^{-3}(0)]^{1/3}} , \qquad (27)$$

where $\beta=3\alpha^2 b_0^2 K^3/4$. The solutions (26) and (27) are the main analytic result of our paper. Together with (25), they give the parameters for each interacting soliton as functions of propagation distance ξ. The solution (25) shows that, during propagation, the larger soliton increases its soliton parameter η_1 and its amplitude A_1 as well and the smaller soliton decreases its parameter η_2 and amplitude A_2. These changes are very slow (the rates of change go to zero in the limit $\varepsilon=\eta_2/\eta_1\rightarrow 0$, as $\partial\eta_{1,2}/\partial\xi\propto\gamma^2\varepsilon^4$). We will discuss the solutions (25)–(27) in more detail in the next section and compare them with the results of our numerical simulations.

IV. NUMERICAL SIMULATIONS

We have carried out numerical simulations of the near-Kerr equation (2) using the standard split-step method with 512 mesh points. We use the initial condition

$$U(\xi=0,\tau)=2\sqrt{2}\,{\rm sech}\tau , \qquad (28)$$

which in the case $\alpha=0$ corresponds to the exact superposition solution

$$U(\xi,\tau)=\frac{4\sqrt{2}[\cosh(3\tau)+3\cosh\tau\exp(i8\xi)]}{\cosh(4\tau)+4\cosh(2\tau)+3\cos(8\xi)}\exp(i\xi) . \qquad (29)$$

For the NLSE this expression gives the result of nonlinear interference between two solitons with initial amplitudes $A_1=3\sqrt{2}$ and $A_2=\sqrt{2}$. We remind the reader that from a near-Kerr system, the soliton parameters η_1 and η_2 are related with a high accuracy to the amplitudes of solitons by simple formulas

$$A_1=\sqrt{2}\eta_1\ , \quad A_2=\sqrt{2}\eta_2\ . \tag{30}$$

We have used absorbers at mesh points close to the boundaries to remove the radiation. The absorption coefficient has been slowly changed from zero to a small value along the mesh to prevent backscattering of the radiation from the boundaries of the absorption region. The total energy loss has been estimated by checking the total energy of the solitons determined by the expression (12). The difference between the energy $Q(\xi)$ at the point ξ and the initial energy $Q(0)$ is equal to the energy lost ΔQ. Radiation appears periodically around the two-soliton solution in the form of separate beams, which are emitted at every period of the beats of the interacting solitons, so that the value $\Delta Q(\xi)$, which we calculate in our simulations, is slightly shifted in ξ relative to the actual energy loss (due to the finite distance between the interaction area and the area of absorption) and is averaged (due to the noninstant absorption).

An example of an inelastic interaction between two solitons in a near-Kerr medium is shown in Fig. 3. The oscillatory behavior of the radiation is clearly seen. Because of the losses due to the radiation, the amplitudes of the two solitons are changing slowly during the interaction process in such a way that the total energy gradually decreases. This process can be seen as internal friction between the two interacting solitons causing radiation. We can calculate the amplitudes and the spatial frequencies of the two solitons in the nonlinear superposition using the fact that the maximum amplitude of the composite solution at each period of the oscillations is equal to

$$A_{\max}=A_1+A_2=\sqrt{2}(\eta_1+\eta_2)\ . \tag{31}$$

The minimum amplitude of the composite solution is

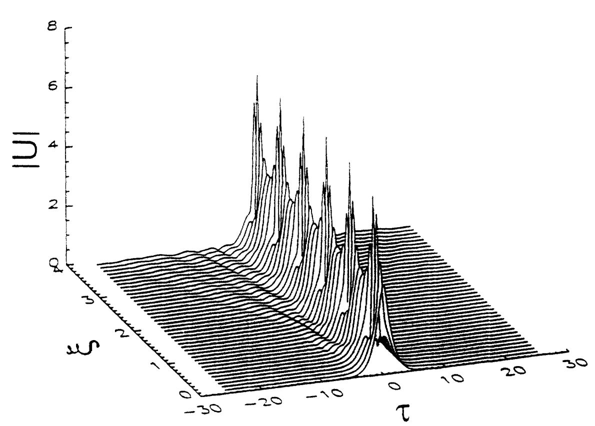

FIG. 3. Perspective plot of the two-soliton interaction in the near-Kerr nonlinear medium with $\alpha=0.005$. The initial condition $U(\xi=0,\tau)$ is defined by the expression (28).

$$A_{\min}=A_1-A_2=\sqrt{2}(\eta_1-\eta_2)\ . \tag{32}$$

The beat frequency is

$$\Omega=\frac{2\pi}{T_\xi}=\eta_1^2-\eta_2^2=\frac{A_{\max}A_{\min}}{2}\ , \tag{33}$$

where T_ξ is the effective period of these oscillations. Hence Eq. (33) can be used to confirm that the amplitudes and the beat frequency are consistent with each other.

We found from our numerical simulations that there is a small transition region at the beginning of the beam propagation $(0<\xi<2)$. The existence of this region is due to the fact that the initial shape (28) corresponds to the two-soliton solution in a Kerr medium and it takes some distance ξ to transform it into a nonlinear superposition of solutions (3). In the transition region, the total energy losses are relatively small in comparison with the energy exchange between the partial solitons (i.e., the solitons which are involved in the nonlinear superposition). After the transition processes have been completed, we can observe the phenomenon of pure internal friction between the two near-Kerr solitons. We present only the results concerning this region of the internal friction.

The amplitude A_2 of the smaller soliton decreases with increasing ξ, but the amplitude of the larger soliton A_1 increases. This means that only the smaller soliton loses energy as a result of internal friction. The larger soliton acquires energy rather than loses it. This in turn means that the energy from the smaller soliton is shared between the energy gained by the larger soliton and the radiation dispersed out of the soliton interaction region. This result is similar to the one which has recently been obtained for the breather interactions in a nonlinear lattice [13]. In multiple random collisions, the energy exchange tends to favor the growth of the larger excitation. In the model used in [13], it was a discreteness-induced phenomenon. In our case, the reason is the explicit deviation from integrability in Eq. (2). We can reformulate the philosophical conclusion of [13] as follows: The world of solitons in non-integrable systems is as merciless for the weak as the real world: the larger solitons grow at the expense of the smaller ones. (Qualitatively this conclusion coincides with the results obtained in [16].)

An example of the A_1 and A_2 versus ξ dependences obtained numerically is shown in Fig. 4. The asymptotic analytic solution (27) is shown in Fig. 4 by the dashed curve. One can see that, in spite of relatively large initial value of $\varepsilon=\eta_2/\eta_1\approx\frac{1}{4}$, our analytic result is in excellent agreement with the numerical one.

If we know the values of A_1 and A_2 (and thus the values of η_1 and η_2), then we can calculate the energies Q_1 and Q_2 using Eq. (6). Although the total energy of the composite soliton solution can be slightly different from the sum of the energies of the solitons (3), this difference is small for a near-Kerr system (when $|\gamma|\ll 1$). This condition determines the range of $|\alpha|$ for our initial condition (28), viz., $|\alpha|\leq 0.005$. The total and differential losses can be calculated from the values of A_1 and A_2 at each step of the simulations. Comparing the differential energy loss of the system of two solitons $-\partial Q/\partial\xi$ with

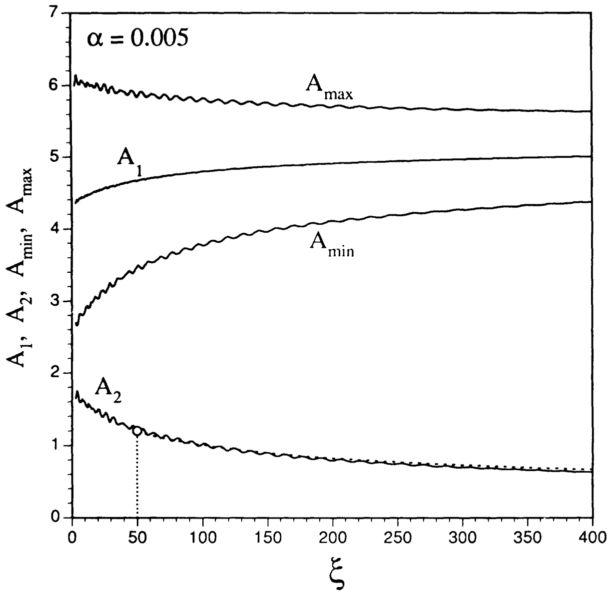

FIG. 4. Amplitudes of the larger (A_1) and the smaller (A_2) solitons versus ξ for $\alpha=0.005$. The analytic result (27) is shown by the dashed curve. The analytic curve starts at $\xi=50$, where $A_1/A_2\approx\frac{1}{4}$. The upper (A_{max}) and the lower (A_{min}) envelopes of the amplitude beats of the two interacting solitons are also presented. (All variables are dimensionless).

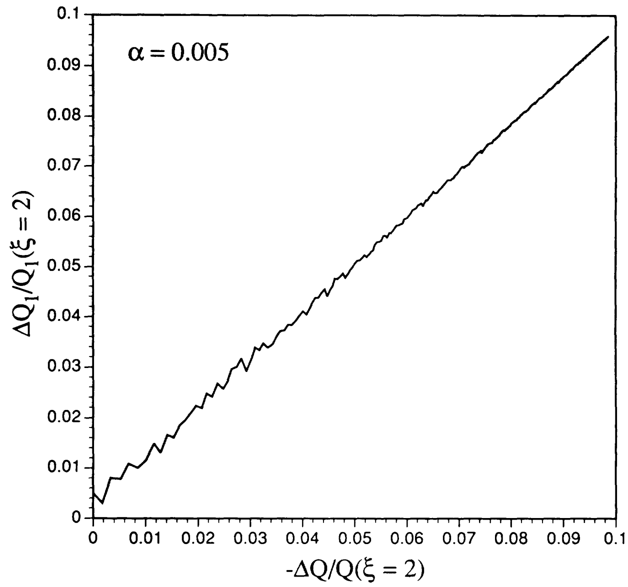

FIG. 5. Normalized energy growth of the larger soliton versus the total energy loss of two interacting solitons for $\alpha=0.005$. The energy losses are normalized to the energy values at $\xi=2$. In this parametric plot, ΔQ and ΔQ_1 are functions of the propagation distance ξ (ξ varies in the range $2\le\xi\le 500$).

the differential increase of energy of the larger soliton $\partial Q_1/\partial\xi$, we have found that the energy lost by the smaller soliton is shared between the radiation and the energy increase of the larger soliton in approximately equal proportions. In other words,

$$\frac{\partial Q_1}{\partial \xi}=-K(\alpha)\frac{\partial Q}{\partial \xi}\ , \tag{34}$$

where K varies in the range 1.0–0.95 when α is changed from 0.0001 to 0.005. Relation (34) is accurate in almost the whole range of propagation distance ($2<\xi<500$), which we have considered (see Fig. 5). The transition region $0<\xi<2$ is excluded. As a result, the energy of the larger soliton has the upper limit

$$Q_{1\,max}=Q_1+\frac{KQ_2}{1+K}\ , \tag{35}$$

where Q_1 and Q_2 are the partial energies of the larger and the smaller solitons, respectively, at any point $\xi>2$.

The difference between the cases $\alpha>0$ and $\alpha<0$ is due to the interaction forces acting between the solitons. If $\alpha>0$, the two solitons attract each other and the energy $Q_{1\,max}$ can be achieved at $\xi=\infty$. If $\alpha<0$, the solitons repel each other and move apart at finite ξ. In this case each soliton has a finite value of energy after the interaction has ceased.

We have checked, using numerical methods, that the same relation between the losses and energy exchange holds when two solitons are colliding. However, the angle of the collision must be small enough so that at least a few beats between the solitons take place when they are interacting. If this condition is fulfilled, the amplitude of the smaller soliton decreases and the amplitude of the larger soliton increases. Radiation is also emitted from the interaction area. At large angles of collision, when the interaction length is shorter than the beat period, the solitons are almost unchanged after a single collision.

The collision of two equal solitons at a small angle can be considered as the interaction of two solitons with equal parameters η_1 and η_2 [see Eq. (8) of [17]]. In this case, the beat period goes to infinity for the Kerr nonlinearity. The inelastic interaction changes the soliton amplitudes, so that the beat period gradually decreases. The process of two solitons merging into one soliton can then be explained as the survival of the soliton with the higher amplitude and the decay of the smaller one. We have observed this process for the case $\alpha>0$. If α is negative, the two solitons repel each other and move apart while the smaller soliton still has energy comparable with the larger one.

V. CONCLUSIONS

In conclusion, we have examined two-soliton inelastic interactions in a near-Kerr system (which is described by the generalized NLSE with a small fifth-order nonlinear susceptibility term). Our analytic investigation and numerical simulations show that not only does radiation appear from the interaction area, but also that the solitons exchange part of their energies during the process. If the angle of a soliton collision is zero (the case of copropagating solitons), then the solitons emit radiation continuously until both of them have enough energy to interact. This process can be considered as internal friction of two solitons. During the interaction, energy is lost only by the soliton with the smaller amplitude. Asymptotically, the amplitude of this smaller soliton decays so that it is inversely proportional to the cube root of the propagation distance. The soliton with the larger amplitude absorbs half of the energy lost by the smaller soliton. The dependences of the amplitudes of each of two interacting solitons as functions of propagation distance ξ are given by the expressions (25)–(27). These analytic results are in excellent agreement with our numerical simulations if the initial ratio of amplitudes of two interacting solitons $\varepsilon=A_2/A_1$ is equal to or less than $\frac{1}{4}$. The cases of nonzero angles of collision have also been considered. The results that we have obtained can be used in estimates of parameter changes during soliton collisions and in experimental observations of the interaction of solitons in various nonlinear media. In general, the analytic approach of this paper can be used for the calculation of radiative losses of two interacting solitons in other systems which are close to integrable.

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APPENDIX

Here we present the analytic solution of Eq. (22) and give the exact value of b_0. The basic equation is

$$\frac{\partial^2 b}{\partial t^2}+b+\frac{4b}{\cosh^2 t}=-\frac{4\sqrt{2}}{\cosh^3 t}+\frac{24\sqrt{2}}{\cosh^5 t}-\frac{24\sqrt{2}}{\cosh^7 t} . \quad (A1)$$

We are interested in a particular solution of Eq. (A1) for the boundary conditions $b(-\infty)=0$ and $(\partial b/\partial t)(-\infty)=0$. First we determine two linearly independent solutions of the corresponding homogeneous equation [i.e., Eq. (A1) with a zero right-hand side part]. These solutions are the associated Legendre functions of the first kind $P_\mu^{\pm i}(z)$, where $z=\tanh(t)$, $i=\sqrt{-1}$, and $\mu=(\sqrt{17}-1)/2$. Now the solution of the inhomogeneous Eq. (A1) [for the boundary conditions $b(-\infty)=0$ and $(\partial b/\partial t(-\infty)=0]$ can be written in the form

$$b(z)=\left|\int_{-1}^{z}\frac{f(z')P_\mu^{-i}(z')}{W[P_\mu^{-i}(z'),P_\mu^{i}(z')]}dz'\right|P_\mu^{i}(z) - \left|\int_{-1}^{z}\frac{f(z')P_\mu^{i}(z')}{W[P_\mu^{-i}(z'),P_\mu^{i}(z')]}dz'\right|P_\mu^{-i}(z) , \quad (A2)$$

where

$$f(z)=-4\sqrt{2}[(1-z^2)^{-1/2}-6(1-z^2)^{1/2}+6(1-z^2)^{3/2}] ,$$

$$W[P_\mu^{-i}(z),P_\mu^{i}(z)]=P_\mu^{-i}(z)[\partial P_\mu^{i}(z)/\partial z] - P_\mu^{i}(z)[\partial P_\mu^{-i}(z)/\partial z]$$

(Wronskian), and $z=\tanh(t)$. The solution (A2) is shown in Fig. 2. The amplitude of the oscillations in the solution (A2), in the limit $t\to\infty$ (or $z\to 1$), gives the value of b_0. All the formulas and integral values that are necessary to calculate this can be found in [18]. The final result is

$$b_0=\frac{4\sqrt{2}\pi^{5/2}}{\cosh(\pi/2)\sqrt{\sinh(\pi)}\Gamma(\delta)\Gamma(\sigma)|\Gamma(\delta+i/2)\Gamma(\sigma+i/2)|} , \quad (A3)$$

where Γ denotes the gamma function, $\delta=(3+\sqrt{17})/4$, and $\sigma=(3-\sqrt{17})/4$. An approximate value of b_0 can also be calculated: $b_0\approx 1.986\,559\,51\ldots$.

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